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An efficient technique for mapping thermal boundary conditions is described and demonstrated. The technique is based on a piece-wise polynomial approximation where the Laplacian derivatives in space are constrained using the heat equation. Measured values for the Laplacian are obtained from temperature rate measurements from sensors embedded within a body. The technique has been implemented in a digital signal processor and is able to provide real-time data on thermal boundary conditions over a surface. The technique is adaptable to complex geometry. In this paper the technique will be applied to a study of the injector-wall interactions in a laboratory scale liquid rocket engine.
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A Finite Element Approach for Multidimensional Inverse Heat Conduction

Edward B. Coy[1] and Matthew Bergkoetter[2], Stephen A. Danczyk[3], Edgar Felix[4]
Air Force Research Laboratory, Edwards AFB,CA,93524

An efficient technique for mapping thermal boundary conditions is described and demonstrated. The technique is based on a piece-wise polynomial approximation where the Laplacian derivatives in space are constrained using the heat equation. Measured values for the Laplacian are obtained from temperature rate measurements from sensors embedded within a body. The technique has been implemented in a digital signal processor and is able to provide real-time data on thermal boundary conditions over a surface. The technique is adaptable to complex geometry. In this paper the technique will be applied to a study of the injector-wall interactions in a laboratory scale liquid rocket engine[5].

Nomenclature

a_i	=	a constant in a polynomial, also a dimension
d	=	vector of data
$\boldsymbol{G}$	=	a matrix of polynomial terms
$\boldsymbol{P}$	=	a vector of polynomials
q	=	heat flux
T	=	temperature
t	=	time
W	=	plate thickness
x,y,z	=	dimensional coordinates
σ	=	a complex constant
ω	=	angular frequency
*	=	complex conjugate
~	=	frequency domain variable
^{T}	=	transpose
$^{-1}$	=	inverse

I. Introduction

Local concentrations of heat flux and temperature are of interest in aerospace applications including injector-wall interactions in liquid rocket engines, shock-shock interactions in external flow over hypersonic vehicles, internal and external flow over blades and vanes in gas-turbine engines, and deposition of directed energy. Accurate characterization of the distributions may be required for validation of a model, optimization of a design, or control of an operational system. There are two basic methods in use today. In the calorimetry approach a heat exchanger is incorporated into the wall of the device and the temperature rise of the working fluid is used to determine the local heat flux. This method is limited to applications where the heat flux is slowly varying in time and changes in one direction only along the surface. Various assumptions and approximations are required to infer the surface temperature; although, in some cases, temperature sensors can be incorporated to give direct measurements. To obtain high spatial resolution with a calorimeter, specialized manufacturing processes are required. The other major class of methods is appropriate for transient situations which commonly occur in hypersonic wind tunnel test articles

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[1] Mechanical Engineer, Aerophysics Branch, 10 East Saturn Blvd., AIAA Member
[2] Physicist, Aerophysics Branch, 10 East Saturn Blvd.
[3] Research Scientist, Aerophysics Branch, 10 East Saturn Blvd., AIAA Member
[4] Mechanical Engineer, Aerophysics Branch, 10 East Saturn Blvd., AIAA Student Member
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and laboratory scale combustion devices where the test articles are designed as heat-sink devices. Surface junction thermocouples have been developed which match the thermal characteristics of the test articles and can be treated as non-intrusive to the flow of heat. From the temperature history a heat flux history can be constructed. In some situations the temperature sensors must be embedded in the wall at some distance from the surface. In this situation, inverse methods must be used to reconstruct the surface heat flux and temperature. The mathematical methods for performing the analysis for surface junction and embedded sensors have been extensively studied and are highly developed; however the most widely used methods assume that the flow of heat is one-dimensional. The potential errors in this approach can be readily appreciated when one considers the situation that occurs when heat is deposited by a concentrated source such as a laser. When energy is deposited adjacent to the sensor, heat diffuses in the plane of the surface causing a temperature rise at the location of the sensor which is recorded incorrectly as heat flux at that point. Due to a plethora of such applications, research in this area has shifted to the development of solution techniques for spatially multi-dimensional and transient problems.

The mathematical characteristics of the inverse heat conduction problem are well known. Changes in boundary conditions cause attenuated and time-lagged responses within a body. In the presence of noise and measurement error, the problem becomes ill-posed and solutions can be biased and unstable. Successful inverse techniques have been developed which rely on the use of information from future time steps to ensure that changes in the boundary conditions at the current time step have had time to reach the sensor locations. Stabilization has also been accomplished through Tikhonov regularization. Other techniques utilize low-order basis functions to dampen instabilities. Most techniques rely on an iterative solution wherein an initial estimate for the boundary condition is made, followed by a forward calculation to establish the effect at the sensor locations, and then a refinement in the boundary condition with repetition until a stopping criteria is satisfied. The need to solve the forward problem multiple times at each time step in a transient analysis results in lengthy run times and must be performed in post-processing. In addition to the destabilizing effects of noise and measurement error and the time required to complete a calculation, practical methods must also contend with the temperature dependence of thermal properties, complex geometries, the need to accurately model the effects of temperature sensor installations, have the ability to handle transient and steady state applications, and be intelligible to those responsible for applying the technique.

II. Theory

In this work we examine the use of the finite element method. Finite element methods are highly developed for forward problems. They are used because the basic approach of subdividing the domain into cells and approximating the solution over each cell with a polynomial is readily adaptable to complex geometries and is mathematically tractable. However, there are some distinct differences between forward and inverse problems that result in some changes in the basic strategy. In solving forward problems it is possible to generate meshes that are limited in size only by the memory capacity of the computer. It is possible to finely discretize regions where the function behavior is complex such that, within each individual cell, the function can be approximated adequately with a simple linear basis function. In inverse problems, as developed here, the grid corresponds to a network of sensors. In practical applications, grids with tens or hundreds of nodes are possible, but probably not thousands, and certainly not the millions that are now common in forward problems. This provides a strong impetus for developing schemes that use high order basis functions. Coy showed how basis functions of arbitrary order can be developed in one spatial dimension using two nodes by constraining the approximating polynomials using the even numbered spatial derivatives from the heat equation. Adding second derivatives improved the frequency response by three orders of magnitude. The model could be formulated as a digital filter and calculated in real time. In this paper we extend the approach to three dimensions.

To fix ideas we consider a set of sensors embedded in a solid material with smoothly varying thermal transport properties. We are primarily interested in constructing approximations for the temperature and heat flux distribution at the surface. The temperature distribution at any point in time within each finite element is approximated by a polynomial.

$$T(x,y,z,t) \approx a_0 + a_1 x + a_2 y + a_3 z + a_4 xy + a_5 xz + a_6 yz + \cdots \tag{1}$$

Or simply as,

$$T(\boldsymbol{x},t) \approx \boldsymbol{P}^{\boldsymbol{T}} \boldsymbol{a} \tag{2}$$

Where,

$$\boldsymbol{P}^T = [1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; \ldots] \tag{3}$$

$$\boldsymbol{a} = [a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \;\; a_6 \;\; \ldots]^T \tag{4}$$

In many applications the placement of the sensors and the choice of approximating polynomial can be optimized based on knowledge of the physical problem at hand. If it is known that gradients in one particular direction are much greater than other directions, the sensor spacing can be adjusted and high order terms in only the direction of the high gradient need to be included. If we assume,

$$\frac{\partial^2 T}{\partial z^2} \gg \frac{\partial^2 T}{\partial x^2} \approx \frac{\partial^2 T}{\partial y^2} \tag{5}$$

The heat equation becomes,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \tag{6}$$

The time derivatives can be obtained from the temperature records at the measurement points. The set of equations to be solved can be written,

$$\begin{bmatrix} \boldsymbol{P}^T(\boldsymbol{x}_1) \\ \vdots \\ \boldsymbol{P}^T(\boldsymbol{x}_n) \\ \dfrac{\partial^2 \boldsymbol{P}^T(\boldsymbol{x}_1)}{\partial z^2} \\ \vdots \\ \dfrac{\partial^2 \boldsymbol{P}^T(\boldsymbol{x}_n)}{\partial z^2} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} T_1 \\ \vdots \\ T_n \\ \dot{T}_1/\alpha_1 \\ \vdots \\ \dot{T}_n/\alpha_n \end{bmatrix} \tag{7}$$

Or simply as,

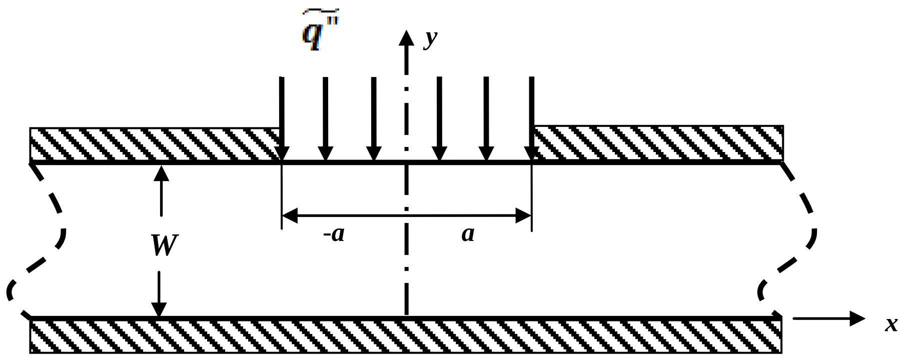
$$\boldsymbol{G}\boldsymbol{a} = \boldsymbol{d} \tag{8}$$

If the number of equations exceeds the number of terms in the polynomial, we can solve as a least-squares problem.

$$\boldsymbol{a} = (\boldsymbol{G}^T \boldsymbol{G})^{-1} \boldsymbol{G}^T \boldsymbol{d} \tag{9}$$

Note that $(\boldsymbol{G}^T \boldsymbol{G})^{-1} \boldsymbol{G}^T$ depends only on the location of the sensors.

The performance of the method can be evaluated by comparing with analytical solutions. Cole has developed Green's function solutions for steady-periodic heating that are suitable for this purpose. Consider a uniform slab of depth W and infinite width with a uniform, single-frequency, steady-periodic heat flux on the surface at y=W, -a<x<a.



The temperature inside the slab can be expressed as,

$$T(x, y, t) = Real\,[\tilde{T}(x, y, \omega)\, e^{j\omega t}] \tag{10}$$

$$\tilde{T}(x, y, \omega) = \frac{\alpha q_0}{k}\left[\frac{1}{W}\int_{-a}^{a}\frac{e^{-\sigma_0|x-x'|}}{2\alpha\sigma_0}dx' + \sum_{n=1}^{\infty}(-1)^n\frac{2}{W}cos\left(\frac{n\pi}{W}y\right)\int_{-a}^{a}\frac{e^{-\sigma_n|x-x'|}}{2\alpha\sigma_n}dx'\right]$$

$$\sigma_n = \left[\left(\frac{n\pi}{W}\right)^2 + j\omega/\alpha\right]^{1/2}$$

$$\int_{-a}^{a}\frac{e^{-\sigma_n|x-x'|}}{2\alpha\sigma_n}dx' = \begin{cases} \dfrac{e^{\sigma_n x}\sinh(a\sigma_n)}{\alpha\sigma_n^2}, x < -a \\ \dfrac{2 - e^{-\sigma_n(a+x)} - e^{-\sigma_n(a-x)}}{2\alpha\sigma_n^2}, -a < x < a \\ \dfrac{e^{-\sigma_n x}\sinh(a\sigma_n)}{\alpha\sigma_n^2}, x > a \end{cases}$$

We will use the analytical function to obtain inputs to the finite element model, and again to compare the amplitude and phase of the analytical function with predictions of the finite element model at points distant from the sensor locations, most importantly at the surface. The model coefficients are real constants that multiply the data vector so we can write,

$$\tilde{\boldsymbol{a}} = (\boldsymbol{G}^T\boldsymbol{G})^{-1}\boldsymbol{G}^T\tilde{\boldsymbol{d}} \tag{11}$$

$$\tilde{\boldsymbol{d}} = \begin{bmatrix} \tilde{T}_1 \\ \vdots \\ \tilde{T}_n \\ j\omega\tilde{T}_1/\alpha_1 \\ \vdots \\ j\omega\tilde{T}_n/\alpha_n \end{bmatrix} \tag{12}$$

The complex temperature predicted by the finite element model can be written,

$$\tilde{T}(\boldsymbol{x}) = \boldsymbol{P}^T\tilde{\boldsymbol{a}} \tag{13}$$

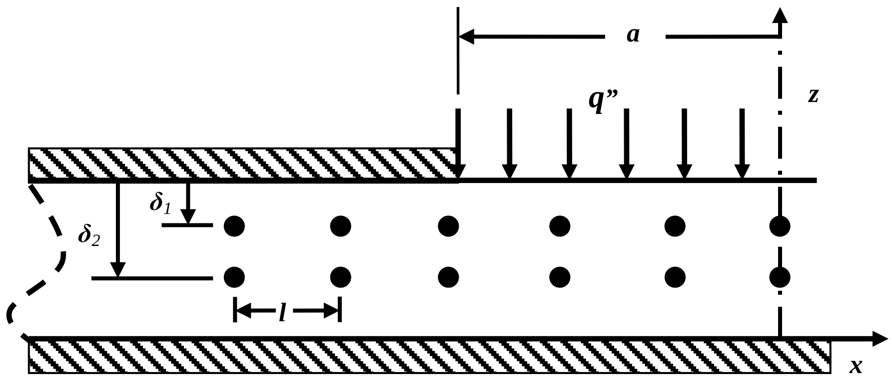
The complex heat flux in the z-direction is,

$$\tilde{q}_{FE}(\boldsymbol{x}) \approx -k\frac{\partial \boldsymbol{P}^T}{\partial z}\tilde{\boldsymbol{a}} \tag{14}$$

The gain and phase-lag of the finite element model relative to the analytical solutions are the following,

$$\Gamma(\boldsymbol{x},\omega) = \frac{[\tilde{T}_{FE}\cdot\tilde{T}^{*}{}_{FE}]^{1/2}}{[\tilde{T}\cdot\tilde{T}^{*}]^{1/2}} \tag{15}$$

$$\varphi(\boldsymbol{x},\omega) = atan[Imag(\tilde{T}_{FE})/Real(\tilde{T}_{FE})] - atan[Imag(\tilde{T})/Real(\tilde{T})] \tag{16}$$



At locations $x = \pm a$ there are gradients of temperature in the x-direction that we would like to capture. We take as a length scale the distance from the heated surface to the first sensor, δ_1. Thus we have, $\delta_1^+ = 1, \delta_2^+ = \delta_2/\delta_1$, $a^+ = a/\delta_1, l^+ = l/\delta_1, W^+ = W/\delta_1, x^+ = x/\delta_1, z^+ = z/\delta_1$. Temperature and frequency are non-dimensionalized as follows: $\tilde{T}^+ = \tilde{T}k/q_0\alpha$, and $\omega^+ = \omega\delta_1^2/\alpha$. We consider a case where a^+ and W^+ are large and look at the effect of frequency, ω^+, on the surface temperature, $\tilde{T}^+$, amplitude and phase for several values of the sensor spacing, l^+.

III. Application

In the final version of this paper we will include information on how we have implemented the technique using a digital signal processor to obtain real-time, video-like imaging of thermal boundary conditions. We will also show an example application in the measurement of thermal boundary conditions in a laboratory scale rocket, showing patterns of heat flux produced by a multi-element injector, and how these patterns change as we deliberately introduce non-uniformities into the flows from the injector elements.